PMM U.S.S.R.,Vol.45,pp.542-549 0021-8928/82/4 0542 $7.50/0

UDC 539.3+532.5

STAMP MOTION ON THE SURFACE OF A THIN COVERING ON A HYDRAULIC FOUNDATION*

V.M. ALEKSANDROV and E.V. KOVALENKO

There is considered the dynamical plane problem of the action of a rigid body (stamp) on an infinitely deep ideal fluid layer. The stamp pressure is transmitted to the fluid through a thin covering (membrane). The stamp moves at a constant velocity over the covering boundary. No friction forces are assumed to exist in the contact domain, and steady potential flow exists in the fluid. Such a problem occurs in an investigation of the process of the dynamical action of solids on an ice cover surface.

The problem is reduced to a one-dimensional integral equation of the first kind of convolution type in a finite interval. The structure of the kernel of the equation obtained, and the characteristic singularities of its solution are studied. It is shown that the problem is correctly solvable only in the class of generalized functions of slow growth. Asymptotic methods are used to construct an approximate solution of the integral equation.

1. Let a deformable layer of slight thickness rest on the surface of an infinitely deep layer of heavy ideal incompressible fluid ($y \leqslant 0$). Furthermore, let a rigid stamp, pressed to the layer by a force P whose eccentricity of application is e, move at a constant velocity V without friction along the boundary of such a composite foundation. We assume that loss of contact of the covering from the fluid does not occur during stamp motion. We also assume that in a moving coordinate system coupled to the stamp, its foundation is described by the function $f(x')$ while the line of contact is defined by the inequality $|x'| \leqslant a$.

The model of a membrane described by the equation

$$-\sigma v'' = p^*(x, t) - q^*(x, t) - \rho^* v^{\cdot\cdot} \tag{1.1}$$

is taken as the physical model of the covering (layer).

Here v is the displacement of membrane points along the y axis, σ is the membrane tension, $p^*(x, t) = p(x')$ is the reactive pressure acting from the fluid on the layer $q^*(x, t) = q(x')$ is the contact pressure which differs from zero only for $|x'| \leqslant a$, $x' = x - Vt$, and ρ^* is the surface density of the covering material.

We will describe the physicomechanical properties of the fluid by linearized steady potential-flow equations

$$\Delta\varphi = 0, \quad v_x = \frac{\partial\varphi}{\partial x'} - V, \quad v_y = \frac{\partial\varphi}{\partial y}, \quad p = \rho V \frac{\partial\varphi}{\partial x'} - \rho g y \tag{1.2}$$

where $\varphi(x', y)$ is the velocity potential, p is the pressure in the fluid, ρ is the fluid density, g is the gravity constant, v_x, v_y are projections of the fluid particle velocity at a point of the flow on the axis of the moving coordinate system.

For $|x'| \leqslant a$ it is also known that because of the condition of contact between the stamp and the covering

$$v = -[\delta + \alpha x' - f(x')] \tag{1.3}$$

Here $\delta + \alpha x'$ is the rigid displacement of the stamp subjected to the applied force P and moment $M = Pe$.

For $y = 0$ the boundary condition (1.1) in the moving coordinate system will have the form

$$-Tv'' = p(x') - q(x'), \quad T = \sigma - \rho^* V^2 \tag{1.4}$$

---

*Prikl.Matem.Mekhan.,45,No.4,734-744,1981

Now, as in thin wing theory /1/, we assume that the condition of fluid contact with the covering surface has the form

$$\frac{\partial\varphi}{\partial y} = -V\frac{\partial v}{\partial x'} \tag{1.5}$$

Then, for $y = 0$ the condition (1.4) and the last relationship in (1.2) can be represented in the form

$$-T\frac{\partial^2\varphi}{\partial x\partial y} = V[p(x) - q(x)] \tag{1.6}$$
$$\frac{\partial p}{\partial x} = \rho V\frac{\partial^2\varphi}{\partial x^2} + \rho g V^{-1}\frac{\partial\varphi}{\partial y}$$

Here and henceforth, we will omit the prime for the moving coordinate x'.

We shall assume that the perturbations the fluid, caused by stamp motion, will vanish as $(x^2 + y^2) \to \infty$.

By using the Fourier integral transform we solve the differential equation (1.2) for φ under the boundary conditions (1.6) and the condition of no perturbations in the fluid at infinity. We obtain the following expression for the displacement v at $y = 0$:

$$v(x) = -\frac{1}{\pi T}\int_{-a}^{a} q(\xi)\,d\xi\int_0^\infty \frac{\cos u(\xi - x)\,du}{u^2 - A_1 u + A_0} \tag{1.7}$$
$$A_1 = \rho V^2 T^{-1}, \quad A_0 = \rho g T^{-1}$$

Let us examine the case when the velocity of stamp motion is $V < V_1$, where $V_1^2 = 2\varkappa^2(\sqrt{1 + V_2^2\varkappa^{-2}} - 1)$, $\varkappa^2 = \rho^* g\rho^{-1}$, and $V_2 = \sqrt{\sigma/\rho^*}$, and is the velocity of transverse wave propagation in the elastic covering. Then the inner integrand in (1.7) has no poles on the real half-axis.

Now using the condition of stamp contact with the covering (1.3), we obtain an integral equation governing the contact pressure distribution law $q(x)$. In dimensionless variables and notation, it will have the following form

$$\int_{-1}^{1}\varphi(\xi')K\left(\frac{\xi' - x'}{\lambda}\right)d\xi' = \pi\lambda^{-1}[\delta' + \alpha x' - r(x')] \quad (|x'| \leqslant 1) \tag{1.8}$$

$$K(z) = \int_0^\infty \frac{\cos uz\,du}{u^2 - 2Bu + 1}, \quad B = \frac{A_1}{2\sqrt{A_0}}, \quad \lambda = \frac{1}{a\sqrt{A_0}}$$

$$\varphi(x') = q(ax')T^{-1}a, \quad r(x') = f(ax')a^{-1}, \quad \delta' = \delta a^{-1}, \quad x' = xa^{-1}$$

We will henceforth omit the primes in (1.8).

Let us note that in the particular case of a stamp at rest ($V = 0$, problem 2), it is necessary to set $B = 0$ in (1.8).

The static condition

$$N_0 = PT^{-1} = \int_{-1}^{1}\varphi(\xi)\,d\xi, \quad N_1 = Pe(Ta)^{-1} = \int_{-1}^{1}\xi\varphi(\xi)\,d\xi \tag{1.9}$$

must still be appended to (1.8).

Moreover, starting from the physical meaning of the problem under consideration, we later require that $v(x) \in C(-R, R)$, where R is an arbitrarily large number. Here $C(-R, R)$ is the space of functions continuous for $|x| \leqslant R$.

2. Let us note that the problem 2 corresponds to a stamp bending a membrane on a Fuss-Winkler foundation with the bedding factor $k = \rho g$. The solution of this problem can be found in closed form by the method of "partition into sections" /2/. To do this, we represent (1.4) with the last relationship in (1.2) taken into account in the form

$$-Tv_1'' + kv_1 = 0 \quad (-\infty < x < -a) \tag{2.1}$$
$$q(x) = Tf''(x) - kv_2, \quad v_2 = -[\delta + \alpha x - f(x)] \quad (|x| \leqslant a)$$
$$-Tv_3'' + kv_3 = 0 \quad (a < x < \infty)$$

It is required to determine $v_i(x)\,(i = 1, 2, 3)$, $q(x)$, the lumped forces P_1, P_2 occuring at the points $x = +a$ and $x = -a$, and also the dependence of the quantities δ and α on the magnitude of the force P and the moment $M = Pe$ applied to the stamp.

The quantity $q(x)$ is determined from the second relationship in (2.1), and from the differential equations of (2.1) under the boundary conditions

$$v_1 \to 0\ (x \to -\infty),\ v_3 \to 0\ (x \to \infty)$$
$$v_1(-a) = -[\delta - \alpha a - f(-a)],\ v_3(a) = -[\delta + \alpha a - f(a)]$$

we find the following expressions for v_1 and v_3:

$$v_{2i+1}(x) = -\{\delta + (-1)^{i+1}\alpha a - f[(-1)^{i+1}a]\} \times \exp\{B_1[a - (-1)^{i+1}x]\},\ B_1^2 = kT^{-1},\quad i = 0, 1 \tag{2.2}$$

We find the lumped forces P_j from the conditions

$$P_1 = T[v_2'(-a) - v_1'(-a)],\ P_2 = T[v_3'(a) - v_2'(a)]$$

Substituting (2.2) here and the formula for v_2 from (2.1), we obtain

$$P_1 = -T[\alpha - f'(-a)] + TB_1[\delta - \alpha a - f(-a)] \tag{2.3}$$
$$P_2 = T[\alpha - f'(v)] + TB_1[\delta + \alpha a - f(a)]$$

Now, using the statics condition

$$P = P_1 + P_2 + \int_{-a}^{a} q(x)\,dx,\quad M = a(P_2 - P_1) + \int_{-a}^{a} xq(x)\,dx$$

we will have

$$P = TB_1\Big[2\delta(aB_1 + 1) - B_1\int_{-a}^{a} f(x)\,dx - f(a) - f(-a)\Big] \tag{2.4}$$
$$M = T\left\{2a\alpha\left(1 + \frac{1}{3}B_1^2a^2 + aB_1\right) - [f(a) - f(-a)](1 + aB_1) - B_1^2\int_{-a}^{a} xf(x)\,dx\right\}$$

Eliminating the quantity δ from the second formula in (2.1) and (2.3) and passing to dimensionless variables and the notation

$$x = ax',\ \lambda = (aB_1)^{-1},\ \varphi^*(x') = aq(ax')T^{-1}$$
$$r(x') = f(ax')a^{-1},\ N_0 = PT^{-1},\ P^* = P_1T^{-1}$$

we arrive in the even case $r(x) = r(-x), \alpha = 0$ at the following relationships (as before, we omit the primes):

$$\varphi^*(x) = r''(x) + \lambda^{-2}[\lambda P^* + \lambda r'(1) + r(1) - r(x)] \tag{2.5}$$
$$P^* = \Big[-(1+\lambda)r'(1) - \lambda^{-1}r(1) + \frac{1}{2}N_0 + \lambda^{-1}\int_0^1 r(x)\,dx\Big](1+\lambda)^{-1}$$

3. We now turn to an investigation of the problems 1 and 2 by using the integral equation (1.8) obtained above. We represent its kernel in the form

$$K(z) = (b_2 - b_1)^{-1}\{\cos(b_1z)\operatorname{ci}(b_1z) + \sin(b_1z) \times [\operatorname{Si}(b_1z) + {}^1/{}_2\pi\operatorname{sgn} z] - \cos(b_2z)\operatorname{ci}(b_2z) - \sin(b_2z) \times [\operatorname{Si}(b_2z) + {}^1/{}_2\pi\operatorname{sgn} z]\},\ b_1 = B - \sqrt{B^2 - 1}\quad b_2 = B + \sqrt{B^2 - 1} \tag{3.1}$$

Here $\operatorname{ci}(x)$, $\operatorname{si}(x)$, and $\operatorname{Si}(x)$ are the integral cosines and sines.

On the basis of (3.1) we can formulate

Lemma 1. For all values of $|z| \leqslant R$, where R is an arbitrarily large number, the following representation is valid

$$K(z) = -\frac{1}{2}\pi|z| + \frac{1}{2}D_1z^2\ln|z| + \frac{1}{2}D_2z^2 + D_3 + F(z) \tag{3.2}$$
$$D_1 = b_1 + b_2,\quad D_2 = D_1\left(C - \frac{3}{2}\right) + \frac{b_2^2\ln b_2 - b_1^2\ln b_1}{b_2 - b_1}$$
$$D_3 = \frac{\ln b_1 - \ln b_2}{b_2 - b_1}$$
$$F(z) \in B_2^1(-R, R),\ F(z) \sim |z|^3\ (z \to 0)$$

Here $B_k^\gamma(-R, R)$ is the space of functions whose k-th derivatives satisfy the Hölder condition with exponent $0 < \gamma \leqslant 1$ for $z \in (-R, R)$, and C is the Euler constant.

Let us investigate the structure of the solution of the integral equation (1.8). To do this we consider the following auxiliary equation

$$\int_{-1}^{1} \varphi(\xi)|\xi - x|\,d\xi = -2\psi(x) \quad (|x| \leqslant 1) \tag{3.3}$$

According to the results in Sect.2, the solution $\varphi(x)$ of the integral equation (3.3) should contain delta-functions at the points $x \pm 1$ in the form of components, which would reflect the appearance of lumped forces at the edges of the line of contact in the contact forces. Meanwhile, as has been noted earlier, according to the physical meaning of the problem under consideration $v(x) \in C(-R, R)$. This condition imposes a constraint on the order of the generalized function $\varphi(x)$. Taking the above into account, we can formulate the following theorems:

Theorem 1. If $\psi''(x) \in C(-1, 1)$, then tne solution of the integral equation (3.3) in the space of generalized functions of slow growth Φ /3/ exists, is inique, and has the form

$$\varphi(x) = -\psi''(x) + P_1\delta(x+1) + P_2\delta(x-1) \tag{3.4}$$

where the constants $P_j\ (j = 1, 2)$ satisfy the relations

$$\begin{gathered}\psi'(1) + \psi'(-1) + P_1 - P_2 = 0 \\ \psi'(-1) - \psi'(1) + \psi(-1) + \psi(1) + P_1 + P_2 = 0\end{gathered} \tag{3.5}$$

($\delta(x)$ is the Dirac delta-function).

Indeed, the function $\varphi(x)$ of the form (3.4) makes the integral equation (3.3) an identity if the relationships (3.5) are satisfied. Uniqueness of the solution (3.4) follows from the theorem presented on p.158 in /3/.

Let us note that for the case of the even function $\psi(x)$ in (3.3), the relationships (3.5) yield together with conditions (1.9)

$$P_1 = P_2 = P^* = \psi'(1) - \psi(1),\ N_0 = -2\psi(1),\ N_1 = 0 \tag{3.6}$$

while for the odd case we have in an analogous manner

$$P_1 = -P_2 = P^* = -\psi'(1),\ \ N_0 = 0,\ \ N_1 = 2\psi(1) \tag{3.7}$$

Now, taking account of the representation (3.2), we rewrite the integral equation (1.8) in the form

$$\begin{gathered}\int_{-1}^{1} \varphi(\xi)\left[|\xi - x| - \frac{D_1}{\pi\lambda}\ln\left|\frac{\xi - x}{\lambda}\right|\right]d\xi = -2\omega(x) \quad (|x| \leqslant 1) \\ \omega(x) = \delta + \alpha x - r(x) - \frac{\lambda}{\pi}\int_{-1}^{1}\varphi(\xi)\left[D_2\frac{(\xi - x)^2}{2\lambda^2} + D_3 + F\left(\frac{\xi - x}{\lambda}\right)\right]d\xi\end{gathered} \tag{3.8}$$

Let us assume that $r''(x) \in C(-1, 1)$. If it is assumed that the function $\varphi(x) \in \Phi$ (with order zero), then because of the properties of the functions $F(z)$ mentioned in the lemma, we will have $\omega''(x) \in C(-1, 1)$. Taking account of Theorem 1, we hence obtain the following assertion.

Theorem 2. If $r''(x) \in C(-1, 1)$, and the solution of the integral equation (3.8) exists in the space of generalized functions of slow growth Φ, then it has the form

$$\varphi(x) = \varphi^*(x) + \frac{D_1}{\pi\lambda}\left[P_1\ln\frac{1+x}{\lambda} + P_2\ln\frac{1-x}{\lambda}\right] + P_1\delta(x+1) + P_2\delta(x-1) \tag{3.9}$$

where the function $\varphi^*(x) \in C(-1, 1)$, and is determined from a Fredholm integral equation of the second kind

$$\begin{gathered}\varphi^*(x) - \frac{1}{\pi\lambda}\int_{-1}^{1}\varphi^*(\xi)\left[D_1\ln\left|\frac{\xi - x}{\lambda}\right| + F''\left(\frac{\xi - x}{\lambda}\right)\right]d\xi = \\ r''(x) + \frac{1}{\pi\lambda}\left(D_2 + \frac{3}{2}D_1\right)N_0 + \frac{1}{\pi\lambda}\left[P_1F''\left(\frac{1+x}{\lambda}\right) + P_2F''\left(\frac{1-x}{\lambda}\right)\right] + \frac{D_1}{(\pi\lambda)^2}[P_1\theta_+(x) + P_2\theta_-(x)] \quad (|x| \leqslant 1) \\ \theta_\pm(x) = \int_{-1}^{1}\ln\frac{1 \pm \xi}{\lambda}\left[D_1\ln\left|\frac{\xi - x}{\lambda}\right| + F''\left(\frac{\xi - x}{\lambda}\right)\right]d\xi\end{gathered} \tag{3.10}$$

The constants $P_j\ (j = 1, 2)$ are connected by the relationships (3.5) and

$$\psi(x) = \omega(x) - \frac{D_1}{2\pi\lambda}\int_{-1}^{1}(\xi - x)^2 \ln\left|\frac{\xi - x}{\lambda}\right| d\xi \tag{3.11}$$

and are determined after having solved (3.10) with (3.9) and (1.9) taken into account. For the even and odd cases of problems 1 and 2, formulas (3.6) and (3.7) also hold.

Considering $\psi(x)$ known temporarily from (3.11), to prove the theorem we turn to the integral operator with kernel $|\xi - x|$ in (3.8). According to (3.4), we obtain the following integral equation of the second kind in $\varphi(x)$:

$$\varphi(x) - \frac{1}{\pi\lambda}\int_{-1}^{1}\varphi(\xi)\left[D_1 \ln\left|\frac{\xi - x}{\lambda}\right| + F''\left(\frac{\xi - x}{\lambda}\right)\right] d\xi = \tag{3.12}$$
$$r''(x) + (\pi\lambda)^{-1}(D_2 + {}^3/_2 D_1)N_0 + P_1\delta(x + 1) + P_2\delta(x - 1)$$
$$(|x| \leqslant 1)$$

We seek the solution of this equation in the form (3.9). Substituting (3.9) into (3.12) and using the properties of the delta function, we arrive at the integral equation (3.10). Let us note that the latter is a Fredholm equation of the second kind with logarithmic kernel and continuous free term. If (3.10) is solvable for a given value of $\lambda \in (0, \infty)$, then the function $\varphi^*(x) \in C(-1, 1)$. The original integral equation (3.8) is here also solvable uniquely.

4. We turn to problem 2 and find its closed solution by using the integral equation (1.8). Taking account of the representation (1.8) for $K(z)$ at $B = 0$, we obtain the integral equation (3.8) to determine $\varphi(x)$, in which we should set $D_1 = 0, D_2 = \pi/2$. As mentioned above, we seek its solution in the form of (3.9) $(D_1 = 0)$. Then the integral equation (3.10) is written in the form

$$\varphi^*(x) - \frac{1}{2\lambda}\int_{-1}^{1}\varphi^*(\xi)\exp\left(-\frac{|\xi - x|}{\lambda}\right) d\xi = r''(x) + \frac{1}{2\lambda}e^{-1/\lambda}(P_1 e^{-x/\lambda} + P_2 e^{x/\lambda}) \quad (|x| \leqslant 1) \tag{4.1}$$

The expression (4.1) will hold under the additional relations (3.5). With the second formula of (3.8), (3.11) $(D_1 = 0)$ and the expression for $F(z)$ taken into account, we write (3.5) in the form

$$2a - r'(1) - r'(-1) + e^{-1/\lambda}\int_{-1}^{1}\varphi^*(\xi)\,\mathrm{sh}\frac{\xi}{\lambda}\, d\xi + {}^1/_2(1 + e^{-2/\lambda})(P_1 - P_2) = 0 \tag{4.2}$$

$$2\delta - r(1) - r(-1) + r'(1) - r'(-1) - (1 + \lambda)e^{-1/\lambda}\int_{-1}^{1}\varphi^*(\xi) \times \mathrm{ch}\frac{\xi}{\lambda}\, d\xi + {}^1/_2[(1 - \lambda) - (1 + \lambda)e^{-2/\lambda}](P_1 + P_2) = 0$$

In the even case $(r(x) = r(-x),\ \alpha = 0)$, by seeking the solution of the integral equation (4.1) under the conditions (4.2) in the form

$$\varphi^*(x) = r''(x) - \lambda^{-1} r(x) + D \quad (D = \text{const})$$

we arrive at (2.5) after a number of manipulations with the statics condition (1.9) taken into account. Here, however, the question of whether the solution constructed is a unique solution of (4.1) occurs. The answer is given in the following theorem.

Theorem 3. The homogeneous equation (4.1) has no positive eigenvalues in the class of functions $\varphi^*(x) \in C(-1, 1) \cap V(-1, 1)$.

Here $V(-1, 1)$ is the space of functions having finite variation on the segment $[-1, 1]$.

For the proof we introduce the Fourier transform of the function $\varphi^*(x)$

$$\Phi^*(u) = \int_{-1}^{1}\varphi^*(\xi)e^{iu\xi} d\xi \quad (\Phi^*(u) = O(u^{-1}),\ |u| \to \infty) \tag{4.3}$$

and we rewrite the homogeneous equation (4.1) as follows

$$\varphi^*(x) - \frac{1}{2\pi\lambda^2}\int_{-\infty}^{\infty}\frac{\Phi^*(u)e^{-iux}}{u^2 + \lambda^{-2}} du = 0 \quad (|x| \leqslant 1) \tag{4.4}$$

Because of the properties of $\varphi^*(x)$. mentioned in the conditions of the theorem, the representation (4.3) is valid, the function $\Phi^*(u)$ is at least continuous, and the estimate of /4/, presented in the parentheses in (4.3) holds.

We multiply both sides of (4.4) by $\varphi^*(x)dx$ and integrate between the limits -1 and +1. Taking account of the Parseval equality /3/, we obtain

$$\int_{-\infty}^{\infty} |\Phi^*(u)|^2 \frac{u^2}{u^2+\lambda^{-2}}\, du = 0 \tag{4.5}$$

It follows from the estimate above that the integral in (4.5) converges. We also note that to satisfy (4.5) it is necessary and sufficient that $\Phi^*(u) \equiv 0$ from which $\varphi^*(x) \equiv 0$ and the theorem is proved.

A similar theorem can be proved analogously for the integral equation (3.10) of problem 1 also.

5. Let us construct asymptotic solutions of the integral equation (3.10) corresponding to problem 1, for large and small values of the parameter $\lambda \in (0, \infty)$.

Taking (3.1) into account, we see that the representation

$$F(z) = (b_2 - b_1)^{-1} \{f_1(b_1 z) \ln b_1 |z| - f_1(b_2 z) \ln b_2 |z| + (2|z|)^{-1}[b_1^{-1} f_2(b_1 z) - b_2^{-1} f_2(b_2 z)] + f_3(b_1 z) - f_3(b_2 z)\} \tag{5.1}$$

$$f_j(z) = \sum_{k=1}^{\infty} a_{jk} z^{2k+2} \quad (j = 1, 2, 3)$$

is valid for $F(z)$ in (3.2) for all $0 \leqslant |z| < \infty$.
The constants a_{jk} are not presented in the interests of brevity.

Following /5/, we seek the solution of the integral equation (3.10) for large values of the parameter λ in the form

$$\varphi^*(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} \varphi_{ij}(x) \lambda^{-i} (\ln \lambda)^j \tag{5.2}$$

Substituting (5.2) into (3.10) and then equating coefficients of identical powers of λ^{-1} and $\ln \lambda$ in the left and right sides of the relation obtained, we obtain an infinite system of recursion relationships for the successive determination of the functions $\varphi_{ij}(x)$:

$$\varphi_{00}(x) = r''(x), \quad \varphi_{10}(x) = \frac{D_1}{\pi} \int_{-1}^{1} \varphi_{00}(\xi) \ln|\xi - x|\, d\xi + \frac{1}{\pi}\left(D_2 + \frac{3}{2} D_1\right)[P_1 + P_2 + r'(1) + r'(-1)], \tag{5.3}$$

$$\varphi_{11}(x) = -\frac{D_1}{\pi} \int_{-1}^{1} \varphi_{00}(\xi)\, d\xi, \ldots$$

Having determined $\varphi_{ij}(x)$ from (5.3), we then find the constants P_j $(j = 1, 2)$, δ and α from (1.9), (3.5) and (3.12) by relating them to N_0 and N_1 and we therefore construct the asymptotic solution of the integral equation of problem 1 for large values of the parameter λ, according to (3.9).

For the even case of the problem, we will have to the accuracy of terms of the order λ^{-2}

$$\varphi(x) = r''(x) + \frac{D_1}{\pi\lambda} \int_{-1}^{1} r''(\xi) \ln\left|\frac{\xi - x}{\lambda}\right| d\xi + P^*\delta(x+1) + P^*\delta(x-1) + \frac{D_1}{\pi\lambda} P^* \ln\frac{1-x^2}{\lambda^2} + \frac{1}{\pi\lambda}(3D_1 + 2D_2)[P^* + r'(1)]$$

$$N_0 = 2[r'(1) + P^*] + \frac{D_1}{\pi\lambda} \int_{-1}^{1} r''(\xi)[(1-\xi)\ln(1-\xi) + (1+\xi)\ln(1+\xi)]\, d\xi + \frac{2}{\pi\lambda}(D_1 + 2D_2)[P^* + r'(1)] + \frac{4}{\pi\lambda} D_1 \ln\frac{2}{\lambda}[P^* + r'(1)]$$

6. We now examine the case of sufficiently small λ. We limit ourselves to the construction of the principal (zero) term of the asymptotic of the solution of problem 1.

We first obtain the degenerate solution of the integral equation (1.8) as $\lambda \to 0$ Making the change of variables $z = y\lambda^{-1}$, $u = \gamma\lambda$ in the kernal $K(z)$ and taking into account that λ is small, and $B \sim \lambda^\circ$, we will have

$$K\left(\frac{y}{\lambda}\right)=\lambda\int_0^\infty \cos\gamma y d\gamma \to \lambda\pi\delta(y)$$

Then the integral equation (1.8) becomes

$$\int_{-1}^{1}\varphi_0(\xi)\delta(\xi-x)d\xi=\lambda^{-2}[\delta+\alpha x-r(x)]\quad (|x|\leqslant 1) \tag{6.1}$$

and is an equation of the contact problem about stamp interaction with a Fuss – Winkler foundation. Its solution can easily be obtained, and has the form

$$\varphi_0(x)=\Omega(x)\lambda^{-2}=\lambda^{-2}[\delta+\alpha x-r(x)] \tag{6.2}$$

We turn to an investigation of (1.8) for $\lambda\ll 1$ by the method of merging asymptotic expansions /6/. We understand the external domain to be the interval $-1+m\lambda\leqslant x\leqslant 1-n\lambda$ on which the "degenerate solution" of the problem (6.2) can be taken as the solution of the equation of problem 1 with sufficiently small error. We call internal domains the small neighborhoods of the points $x=\pm 1$ with size $n\lambda$ and $m\lambda$; in these domains the influence of the covering on the contact stress distribution under the stamp is commensurate with the quantity $\varphi_0(x)$. Boundary-layer type solutions should be constructed in the internal domains, which should reflect the characteristic behavior of the function $\varphi(x)$ in the neighborhoods of the points $x=\pm 1$, and should smoothly merge with the degenerate solution $\varphi_0(x)$ on the domain boundaries $x=1-n\lambda$, $x=-1+m\lambda$ in conformity with (3.9).

Taking into account that for $n\lambda\ll 1$ and $m\lambda\ll 1$

$$\begin{aligned}\varphi_0(1-n\lambda)&=\Omega(1)\lambda^{-2}[1+O(n\lambda)]\\ \varphi_0(-1+m\lambda)&=\Omega(-1)\lambda^{-2}[1+O(m\lambda)]\end{aligned} \tag{6.3}$$

we will see a solution of boundary-layer type in the neighborhood of the points $x=\pm 1$ in the respective forms

$$\begin{aligned}\varphi_+(x)&=\lambda^{-2}\chi\left(\frac{1-x}{\lambda}\right)[1+O(n\lambda)]\\ \varphi_-(x)&=\lambda^{-2}\Psi\left(\frac{1+x}{\lambda}\right)[1+O(m\lambda)]\end{aligned} \tag{6.4}$$

We shall perform the merger by considering that for any sufficiently large m and n:

$$\varphi_0(1-n\lambda)=\varphi_+(1-n\lambda),\ \varphi_0(-1+m\lambda)=\varphi_-(-1+m\lambda) \tag{6.5}$$

In order to obtain an equation to determine the function $\chi(t)$, we substitute $\varphi_+(x)$ from (6.4) into the integral equation (1.8), make the change of variables of the form $t=(1-x)\lambda^{-1}$, $\tau=(1-\xi)\lambda^{-1}$ and then let λ tend to zero. We consequently arrive at the deduction that the function $\chi(t)$ should be found from the integral equation

$$\int_0^\infty \chi(\tau)K(t-\tau)d\tau=\pi\Omega(1)\quad (0\leqslant t<\infty) \tag{6.6}$$

We see in an analogous manner that the function $\Psi(t)$ $(t=(1+x)\lambda^{-1})$ should also satisfy an equation of the type (6.6).

By using (6.2)—(6.6), we determine the uniformly-suitable /6/ asymptotic solution of the integral equation (1.8)

$$\varphi_u(x)=\Lambda(x)+P_1\delta(x+1)+P_2\delta(x-1)=\lambda^{-2}\left[\Omega(x)+\chi\left(\frac{1-x}{\lambda}\right)+\Psi\left(\frac{1+x}{\lambda}\right)-\Omega(1)-(\Omega-1)\right] \tag{6.7}$$

7. Therefore, there remains to find the solution of the Wiener – Hopf integral equation (6.6). As is known /7/, the main difficulty is in factorizing the function

$$L_\varepsilon(u)=(u^2-2B\sqrt{u^2+\varepsilon^2}+1)^{-1}=L_+(u)L_-(u)\quad (\varepsilon\to 0)$$

hence by using the idea of Koiter /7/, we approximate $L(u)$ by the following:

$$L(u)=\frac{1}{u^2+1}\frac{P_1^*(u)}{P_2^*(u)} \tag{7.1}$$

where $P_1^*(u), P_2^*(u)$ are even polynomials in identical orders $2n$ without zeroes on the real axis.

Furthermore, limiting ourselves to obtaining just qualitative results, we examine the simplest case of (7.1) by setting $P_1^*(u)=P_2^*(u)=1$. In conformity with (3.9), we seek the solution of the integral equation (6.6) in the form

$$\chi(t) = \chi^*(t) + Q_2\delta(t) \qquad (Q_2 = \lambda P_2) \tag{7.2}$$

Substituting (7.2) into (6.6), we will have

$$\int_0^\infty \chi^*(\tau) K(t-\tau)\, d\tau = \pi[\Omega(1) - \pi^{-1}Q_2K(t)] \quad (0 \leqslant t < \infty) \tag{7.3}$$

from which we obtain the following Riemann problem for the real axis by the methods in /8/:

$$\chi_+^*(\alpha) = (1+\alpha^2)\chi_-^*(\alpha) + \frac{i}{\sqrt{2\pi}}(1+\alpha^2)\left[\frac{\Omega(1)}{\alpha} - \frac{Q_2}{2}\frac{1}{i+\alpha}\right] \tag{7.4}$$

As is known /9/, because of the completely definite selection of the constant Q_2, all the solvability conditions introduced in /8/ for the functional equation (7.4) can be avoided in the class of functions decreasing at infinity. Namely, by setting

$$Q_2 = \Omega(1) \tag{7.5}$$

we find $\chi^*(t) = \Omega(1)$. We note that the first of the merger conditions (6.5) is satisfied here. We also note that (7.5) is a solvability condition of the kind (3.5) for (1.8) as $\lambda \to 0$ in the space of generalized functions of slow growth.

A solution of boundary-layer type can analogously be constructed on the edge $x = -1$.

The relation between the quantities N_0, N_1 and δ, α is found from the statics condition (1.9) with (7.2), (7.5), (6.2), (6.5) taken into account.

In the case of a stamp with rounded corners, the half-length a of the contact line becomes an unknown. To determine the quantity a in this case, it should be taken into account that the function $v^\circ(x) \in C(-R, R)$, and hence, $P_1 = P_2 = 0$. This condition is an additional condition for the determination of a.

In conclusion, let us note that the problem of motion of a rigid stamp over a Kirchhoff–Love plate lying on a hydraulic base can be studied in an analogous way. Here the general solution of the problem will contain the expression

$$P_1\delta(x+1) + M_1\delta'(x+1) + P_2\delta(x-1) + M_2\delta'(x-1)$$

as a component.

The authors are grateful to N.Kh. Arutiunian for valuable remarks.

REFERENCES

1. SEDOV L.I., Plane Problems of Hydrodynamics and Aerodynamics. NAUKA, Moscow, 1966.
2. ALEKSANDROV V.M., Certain contact problems for beams, plates, and shells. Inzh. Zh., Vol.5, No.4, 1965.
3. VLADIMIROV V.S., Equations of Mathematical Physics. NAUKA, Moscow, 1976.
4. BOKHNER S., Lectures on Fourier Integrals. FIZMATGIZ, Moscow, 1962.
5. ALEKSANDROV V.M. and ARUTIUNIAN N.Kh., Interaction of a moving elastic stamp with an elastic half-plane through a stiffener or a thin ideal fluid layer, PMM, Vol.42, No.3, 1978.
6. VAN DYKE M., Perturbation Methods in Fluid Mechanics. Academic Press, N.Y. and London, 1964.
7. NOBLE B., Methods Based on the Wiener–Hopf Technique for the Solution of Partial Differential Equations, Pergamon Press, 1959.
8. GAKHOV F.D., and CHERSKII Iu.I., Equations of Convolution Type. NAUKA, Moscow, 1978.
9. IVANOV V.V., On Wiener–Hopf equations of the first kind. Dokl. Akad. Nauk SSSR, Vol.151, No.3, 1963.

Translated by M.D.F.